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Singular Invariant Tempered Distributions on Regular Prehomogeneous Vector Spaces

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INTRODUCTION

Let $P(x)$ be an irreducible homogeneous polynomial with real coefficients on a real vector space V. We suppose that $S=\{x \in V; P(x)=0\}$ decomposes into a finite number of G^1-orbits with $G^1=\{g \in GL(V); P(g \cdot x)=P(x)\}$. We put $V_1 \cup V_2 \cup \cdots \cup V_l = V-S$ be the connected component decomposition. Then, is any G^1-invariant tempered distribution supported in S obtained as a finite linear combination of negative Laurent coefficients of the tempered distribution $|P(x)|^s|_{V_i}$? The purpose of this paper is to give a necessary and sufficient condition for this problem.

We shall explain our problem more precisely. Let $P(x)$ be an irreducible homogeneous polynomial of degree d on a finite dimensional complex vector space $V_{\mathbb{C}}$ satisfying the condition that the determinant of the Hessian $\det(\partial P(x)/\partial x_i \partial x_j)$ does not identically vanish on $V_{\mathbb{C}}$, and let $G^1_{\mathbb{C}}$ be a rational closed connected subgroup of $GL(V_{\mathbb{C}})$ contained in the subgroup $\{g \in GL(V_{\mathbb{C}}); P(g \cdot x)=P(x)\}$. We suppose that:

(A) (1) *The set $S_{\mathbb{C}}=\{x \in V_{\mathbb{C}};\ P(x)=0\}$ decomposes into a finite number of $G^1_{\mathbb{C}}$-orbits.*

(2) *The open set $V_{\mathbb{C}}$-$S_{\mathbb{C}}$ is a single $G^1_{\mathbb{C}} \times GL_1(\mathbb{C})$-orbit.*

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Here, $GL_1(\mathbb{C})$ acts on $V_{\mathbb{C}}$ by constant multiplication. Then the pair $(G^1_{\mathbb{C}} \times GL_1(\mathbb{C}), V_{\mathbb{C}})$ is a regular prehomogeneous vector space. We denote by $G_{\mathbb{C}}$ the algebraic group $G^1_{\mathbb{C}} \times GL_1(\mathbb{C})$. We say that $S_{\mathbb{C}}$ is the *singular set* of the prehomogeneous vector space.

Let $(G^1 \times GL_1(\mathbb{R})^+, V)$ be a real form of $(G^1_{\mathbb{C}} \times GL_1(\mathbb{C}), V_{\mathbb{C}})$. Here, we put G^1 the connected component subgroup of $\{g \in G^1_{\mathbb{C}}; g \in GL(V)\}$ with the neutral element. We denote by $G^+_{\mathbb{R}}$ the group $G^1 \times GL_1(\mathbb{R})^+$. Let $V_1 \cup \cdots \cup V_l$ be the connected component decomposition of the set $V - S$. Then each V_i is a $G^1 \times GL_1(\mathbb{R})^+$-orbit. We suppose that:

(A) (3) *The restriction of $P(x)$ on V is a polynomial with real coefficients.*

Consider the integral

$$\Phi_i(s, f) = \int_{V_i} |P(x)|^s f(x)\, dx, \tag{0.1}$$

with $f(x) \in \mathscr{S}(V)$, $s \in \mathbb{C}$ and dx a Euclidean measure on V. Here $\mathscr{S}(V)$ is the Schwartz space of rapidly decreasing functions on V. Then $\Phi_i(s, f)$ converges absolutely for any $f \in \mathscr{S}(V)$, and the linear functional

$$f \mapsto \Phi_i(s, f) \tag{0.2}$$

defines a tempered distribution on V with a holomorphic parameter $s \in \mathbb{C}$ if the real part of s is sufficiently large. From the general results of the theory of b-functions for the complex power $P(x)^s$, (0.1) can be continued meromorphically to the whole complex plane $s \in \mathbb{C}$ (see [Sm-Sh]), and the set of the locations of poles is a subset of the set of the roots of the b-function and they are negative rational numbers (Kashiwara's theorem; see [K1]). We denote by $|P(x)|^s_i$ the tempered distribution thus obtained.

From the definition, the tempered distributions $|P(x)|^s_i$ is a $G^+_{\mathbb{R}}$-relatively invariant tempered distribution corresponding to the character $\chi(g)^s = g_2^{ds}$ with $g = (g_1, g_2) \in G^+_{\mathbb{R}}$, i.e., $|P(\rho(g)\, x)|^s_i = \chi(g)^s\, |P(x)|^s_i$ for any $g \in G^+_{\mathbb{R}}$. Let λ_k $(k = 0, 1, 2, ...)$ be a point in $\mathbb{C}$ where $\Phi_i(s, f)$ has a pole. We have the Laurent expansion at $s = \lambda_k$,

$$|P(x)|^s_i = \sum_{-j_k \leqslant j < +\infty} T_i{}^j{}_k(x) \cdot (s - \lambda_k)^j, \tag{0.3}$$

where j_k is the order of the pole at $s = \lambda_k$. Then it is easy to see that $T_i{}^j{}_k(x)$ is a G^1-invariant tempered distribution whose support is contained in the set S if $j < 0$. For a tempered distribution $T(x)$ on V, we call $T(x)$ a *singular tempered distribution* if the support of $T(x)$ is contained in the set S. That is, each $T_i{}^j{}_k(f)$ is a G^1-invariant singular tempered distribution. Our problem is the converse of this fact.

MAIN PROBLEM. *Under the condition* (A), *is any singular G^1-invariant tempered distribution obtained as a finite linear combination of $T_i{}^j{}_k(x)$ in* (0.3) *with* $1 \leqslant i \leqslant l$, $k = 0, 1, 2, \ldots$, *and* $j_k \leqslant j < 0$?

In this paper, we give a necessary and sufficient condition for this problem by using the theory of holonomic systems (Theorem 2.7).

THEOREM 2.7. *The following conditions* (B) *and* (C) *are equivalent under the condition* (A).

(B) *Any G^1-invariant singular tempered distribution is obtained as a finite linear combination of $T_i{}^j{}_k(x)$ in* (0.3) *with* $1 \leqslant i \leqslant l$, $k = 1, 2, \ldots$, *and* $j_k \leqslant j < 0$.

(C) *Let $\lambda \in \mathbb{C}$. Any relatively invariant tempered distribution $T(x)$ corresponding to the character $\chi(g)^\lambda$ is obtained as a linear combination*:

$$T(x) = \sum_{i=1}^{l} a_i(s) \cdot |P(x)|_i^s \,|_{s=\lambda}, \tag{0.4}$$

with holomorphic functions $a_i(s)$ at $s = \lambda$.

Our proof is based on the finite dimensionality theorem (Proposition 2.2) for holonomic systems and is rather different from the methods that have been used, for example, by Raïs [Ra] or Rubenthaler [Ru1].

It may seem to be difficult to verify the condition (C) in order to show (B). However, it is not difficult to show that (C) is true if we may use the micro-local analysis. For example, we can show that (C) is actually valid when $P(x)$ is a relatively invariant polynomial of prehomogeneous vector spaces of "commutative parabolic type" introduced by Muller, Rubenthaler, and Schiffmann [Mu-Ru-Sc]. They contain the cases of Raïs [Ra] and E. M. Stein [St] as examples. The proof of the verification of (C) in such cases will appear in a future paper [Mr2]. In particular, the author has carried out detailed computations on an invariant measure on a singular orbit of prehomogeneous vector spaces of commutative parabolic type. See [Mr1]. Wright [Wr] and Datscovsky and Wright [Da-Wr] dealt with the singular invariant distributions on the space of binary cubic forms. Our theorem is applied to this case. Moreover, Weil [We] proposed the complete determination problem of invariant tempered distributions in connection with the generalization of Tate's work. Our theorem would contribute to such problem.

Notation. We denote by $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}^+$ the sets of complex numbers, real numbers, and positive integers, respectively. The space $\mathscr{S}(V)$ is the Schwartz space of rapidly decreasing functions on the vector space V. We denote by $GL(V)$ the group of invertible linear endomorphisms on the vector space V. In particular, when $V = K^n$ with a field K, $GL(V)$ is denoted by $GL_n(K)$.

1. Preliminaries

1.1. *Reviews of Prehomogeneous Vector Spaces*

We begin with a review of basic notions of prehomogeneous vector spaces. Let $G_{\mathbb{C}}$ be a complex connected linear algebraic group, $V_{\mathbb{C}}$ a finite dimensional complex vector space and ρ a rational linear representation of $G_{\mathbb{C}}$ on $V_{\mathbb{C}}$. We say that the triplet $(G_{\mathbb{C}}, \rho, V_{\mathbb{C}})$ is a *prehomogeneous vector space* if there exists a Zariski open dense $G_{\mathbb{C}}$-orbit in $V_{\mathbb{C}}$. Such an open dense orbit is unique. We call it an *open orbit* or a *big orbit*. Orbits other than the open one are called *singular orbits* or *small orbits*. Let $P(x)$ be a polynomial on $V_{\mathbb{C}}$. We say that $P(x)$ is a *relatively invariant polynomial*, or simply *relative invariant*, if there exists a character $\chi(g)$ of $G_{\mathbb{C}}$ satisfying

$$P(\rho(g)\cdot x) = \chi(g)\, P(x), \tag{1.1}$$

for any $g \in G_{\mathbb{C}}$. We call $\chi(g)$ the *corresponding character* of $P(x)$. We say that $(G_{\mathbb{C}}, \rho, V_{\mathbb{C}})$ is *regular* if there exists a relative invariant $P(x)$ satisfying that the determinant of the Hessian $\det(\partial P/\partial x_i\, \partial x_j)$ does not identically vanish on $V_{\mathbb{C}}$.

On the other hand, we say that a homogeneous polynomial $P(x)$ is *non-degenerate* if the determinant of its Hessian $\det(\partial P(x)/\partial x_i\, \partial x_j)$ does not identically vanish on $V_{\mathbb{C}}$. Now, conversely, we suppose that a non-degenerate irreducible homogeneous polynomial $P(x)$ on a complex vector space $V_{\mathbb{C}}$ has been given. We let

$$S_{\mathbb{C}} = \{x \in V_{\mathbb{C}}; P(x) = 0\}. \tag{1.2}$$

Let $G^1_{\mathbb{C}}$ be a linear algebraic subgroup $\{g \in GL(V_{\mathbb{C}}); P(g\cdot x) = P(x)\}$ of $GL(V_{\mathbb{C}})$. We suppose that:

(A) (1) *The set $S_{\mathbb{C}}$ decomposes into a finite number of $G^1_{\mathbb{C}}$-orbits.*

(2) *The set $V_{\mathbb{C}} - S_{\mathbb{C}}$ is a single $G^1_{\mathbb{C}} \times GL_1(\mathbb{C})$-orbit.*

Here, $GL_1(\mathbb{C})$ acts on $V_{\mathbb{C}}$ as constant multiplication. The group $G^1_{\mathbb{C}} \times GL_1(\mathbb{C})$ is naturally a subgroup of $GL(V_{\mathbb{C}})$ and we denote it by $G_{\mathbb{C}}$. The linear representation of $G_{\mathbb{C}}$ on $V_{\mathbb{C}}$ is denoted by $\rho(g)$ with $g \in G_{\mathbb{C}}$. If the condition (A)(2) is satisfied, then the triplet $(G_{\mathbb{C}}, \rho, V_{\mathbb{C}})$ is a regular prehomogeneous vector space from the definition. The polynomial $P(x)$ is naturally a relative invariant of the prehomogeneous vector space $(G_{\mathbb{C}}, \rho, V_{\mathbb{C}})$. We denote by $\chi(g)$ the character corresponding to the polynomial $P(x)$. In this case, $\chi(g) = g_2{}^d$ with d = degree of $P(x)$ and $g = (g_1, g_2) \in G^1_{\mathbb{C}} \times GL_1(\mathbb{C})$. In particular, from the definition, we have

$$G^1_{\mathbb{C}} = \{ g \in G_{\mathbb{C}} ; \chi(g) = 1 \}. \tag{1.3}$$

The subset $S_{\mathbb{C}}$ is called a *complex singular set*, or simply, a *singular set*.

In this paper, we always deal with a polynomial $P(x)$ whose associated invariant complex algebraic group $G^1_{\mathbb{C}}$ satisfies the condition (A). Or we may say that we always consider the prehomogeneous vector space $(G_{\mathbb{C}}, \rho, V_{\mathbb{C}})$ with a relative invariant $P(x)$ satisfying the condition (A). At any rate, we shall consider the problem given in the Introduction only for polynomials that satisfy the condition (A).

Next, we consider the b-function of $P(x)$. Let ρ^* be the contragredient representation of ρ on the dual vector space $V^*_{\mathbb{C}}$. We denote by $\langle x, y \rangle$ the canonical bilinear form on $(x, y) \in V_{\mathbb{C}} \times V^*_{\mathbb{C}}$. Then $(G_{\mathbb{C}}, \rho^*, V^*_{\mathbb{C}})$ is also a regular prehomogeneous vector space with an irreducible relative invariant $Q(y)$ whose corresponding character is $\chi^{-1}(g)$, i.e., $Q(\rho^*(g) \cdot y) = \chi^{-1}(g)\, Q(y)$. The degree of $Q(y)$ coincides with that of $P(x)$. Then we have

$$Q\left(\frac{\partial}{\partial x}\right) \cdot P(x)^{s+1} = b(s) \cdot P(x)^s, \tag{1.4}$$

where $b(s)$ is a polynomial in $s \in \mathbb{C}$. We call $b(s)$ the *b-function* for $P(x)$, (see [Sm-Sh]). The roots of $b(s) = 0$ are all negative rational numbers [K1]. By decomposing $b(s)$ into the product of prime divisors, we have

$$b(s) = c \cdot (s + s_1)(s + s_2) \cdots (s + s_d), \tag{1.5}$$

with $0 < s_1 \leqslant s_2 \leqslant \cdots \leqslant s_d$, where d is the degree of $P(x)$ and c is a constant. The explicit computation of b-function for various examples has been given in [Ki].

1.2. *Invariant Tempered Distributions*

We shall consider real forms of prehomogeneous vector space. Let $G_{\mathbb{R}}$ be a real form of the complex algebraic group $G_{\mathbb{C}}$ and let $G^+_{\mathbb{R}}$ be the connected component of $G_{\mathbb{R}}$ containing the neutral element. Let V be a real form of the complex vector space $V_{\mathbb{C}}$. We say that $(G^+_{\mathbb{R}}, \rho, V)$ is a *real form* of $(G_{\mathbb{C}}, \rho, V_{\mathbb{C}})$ if $\rho(G^+_{\mathbb{R}})$ is the connected component of the group $\rho(G_{\mathbb{C}}) \cap GL(V)$. There may exist several real forms for one complex prehomogeneous vector space. We take one of them, fix it, and denote it by $(G_{\mathbb{R}}^{+}, \rho, V)$. We suppose that:

(A) (3) *The restriction of $P(x)$ on V is a polynomial with real coefficients.*

Remark. As proved in [Sm-Sh], it is always possible to choose a polynomial $P(x)$ such that it is with real coefficients if we do not suppose the irreducibility of $P(x)$. So the condition (A)(3) may not be necessary in that case.

We let

$$S = S_{\mathbb{C}} \cap V = \{x \in V; P(x) = 0\}. \tag{1.6}$$

We call S the *real singular set* of $(G_{\mathbb{R}}^{+}, \rho, V)$, or simply, the *singular set*. Let

$$V_1 \cup V_2 \cup \cdots \cup V_l = V - S \tag{1.7}$$

be the connected component decomposition. From the condition (A)(2), each V_i is a $G_{\mathbb{R}}^{+}$-orbit. We put $G^1 = G_{\mathbb{C}}^1 \cap G_{\mathbb{R}}^{+}$. From the assumption (A)(1), it is clear that the set S decomposes into a finite number of G^1-orbits.

We shall consider some kinds of invariant tempered distributions on the vector space V in the following. From the condition (A)(3), $\chi(g)$ is a positive real valued function on $G_{\mathbb{R}}^{+}$. Therefore, $\chi(g)^s$ is well defined for any $s \in \mathbb{C}$, and it is a character of $G_{\mathbb{R}}^{+}$. Let $T(x)$ be a tempered distribution on V. We say that $T(x)$ is a *relatively invariant tempered distribution corresponding to the character* $\chi(g)^s$ with $s \in \mathbb{C}$ if $T(x)$ satisfies

$$T(\rho(g) \cdot x) = \chi(g)^s \cdot T(x), \tag{1.8}$$

for any $g \in G_{\mathbb{R}}^{+}$. We say that $T(x)$ is a *singular invariant tempered distribution* on V if $T(x)$ satisfies

$$T(\rho(g) \cdot x) = T(x), \qquad \text{for any } g \in G^1, \tag{1.9}$$

and if the support of $T(x)$, which we denote by $\operatorname{supp}(T(x))$, is contained in the set S.

1.3. *Tempered Distributions with a Meromorphic Parameter*

We define a tempered distribution with a holomorphic, or a meromorphic, parameter $s \in \mathbb{C}$ for later use. Let Ω be a domain in $\mathbb{C}$. Let

$$s \mapsto u(s, x)$$

be a map from Ω to the space $\mathscr{S}'(V)$ of tempered distributions on V. We say that $u(s, x)$ is *a tempered distribution on V with a holomorphic* *parameter* s *in* Ω, or simply, *holomorphic with respect to* $s \in \Omega$, if it satisfies the condition that

$$\text{For any } f(x) \in \mathcal{S}(V), \text{ the integral } \int u(s, x) f(x)\, dx \text{ is a holomorphic function in } s \in \Omega. \tag{1.10}$$

Let $u(s, x)$ be a tempered distribution with a holomorphic parameter $s \in \Omega - D$, where D is a discrete subset in Ω. If there exists a holomorphic function $a(s)$ in Ω such that $a(s) \cdot u(s, x)$ is continued as a tempered distribution with a holomorphic parameter s in Ω, then we say that $u(s, x)$ is a *tempered distribution with a meromorphic parameter* s *in* Ω. We say that $u(s, x)$ has *a pole* at $s = \lambda$ if $u(s, x)$ is not holomorphic with respect to s at $s = \lambda$. If $(s - \lambda)^j \cdot u(s, x)$ is holomorphic with respect to s near $s = \lambda$ for a positive integer j and if $(s - \lambda)^{j-1} \cdot u(s, x)$ is not holomorphic, then we say that $u(s, x)$ has a *pole of order* j *at* $s = \lambda$. The following properties are easily proved, so we omit the proofs.

PROPOSITION 1.1. *Let* $u(s, x)$ *be a tempered distribution on* V *with a holomorphic* (*resp. meromorphic*) *parameter* s *in a domain* Ω. *Then we have*:

(1) *The derivative* $(\partial/\partial s)\, u(s, x)$ *of* $u(s, x)$ *with respect to* $s \in \Omega$ *is a tempered distribution with a holomorphic* (*resp. meromorphic*) *parameter* $s \in \Omega$. *Here, the derivative of* $u(s, x)$ *stands for the linear functional on* $\mathcal{S}(V)$ *given by*

$$f(x) \mapsto \frac{\partial}{\partial s} \int u(s, x) f(x)\, dx, \qquad (f(x) \in \mathcal{S}(V)). \tag{1.11}$$

(2) (*Taylor expansion*) *Assume that* $u(s, x)$ *is holomorphic with respect to* s *in a neighborhood of* $s_0 \in \Omega$. *Then there exists a positive number* ε *such that we have the Taylor expansion of* $u(s, x)$;

$$u(s, x) = \sum_{j=0}^{\infty} (s - s_0)^j \cdot T_j(x), \tag{1.12}$$

on $D_{s_0,\varepsilon} = \{s \in \mathbb{C};\ |s - s_0| < \varepsilon\}$ *with* $T_j(x) = (1/j!)(\partial/\partial s)^j\, u(s, x)|_{s=s_0}$ $(j = 0, 1, \cdots)$. *The right-hand side of* (1.12) *is uniformly convergent on* $D_{s_0,\varepsilon}$ *as a tempered distribution on* V.

(3) (*Laurent expansion*) *Assume that* $u(s, x)$ *has a pole of order* j_0 *at* $s = s_0$ *with* $s_0 \in \Omega$. *Then there exists a positive number* ε *such that we have the Laurent expansion of* $u(s, x)$;

$$u(s, x) = \sum_{j=0}^{\infty} (s - s_0)^{j - j_0} \cdot T_j(x), \tag{1.13}$$

on $D_{s_0,\varepsilon}$ *with* $T_j(x) = (1/j!)(\partial/\partial s)^j\,((s-s_0)^{j_0}\cdot u(s,x))\,|_{s=s_0}$ $(j=0,1,\cdots)$. *The right-hand side of* (1.13) *is uniformly convergent on* $D_{s_0,\varepsilon} - D_{s_0,\varepsilon'}$ *for any* $\varepsilon' < \varepsilon$.

(4) (*Analytic continuation*) *Let* $v(s,x)$ *be another tempered distribution with a meromorphic parameter* $s \in \Omega$. *If* $v(s,x)$ *coincides with* $u(s,x)$ *in a neighborhood of a point* $s_0 \in \Omega$, *then they coincide with each other on the whole domain* Ω.

1.4. *Complex Powers of Relative Invariants*

The tempered distributions with a meromorphic parameter $s \in \mathbb{C}$ that we shall deal with in this paper are as follows. Consider the functions

$$|P(x)|_i^s = \begin{cases} |P(x)|^s, & \text{if } x \in V_i, \\ 0, & \text{if } x \notin V_i, \end{cases} \tag{1.14}$$

for a complex number s whose real part, which we denote by $\mathrm{Re}(s)$, is sufficiently large. Then $|P(x)|_i^s$ is a continuous homogeneous function on V. Therefore it can be viewed as a tempered distribution on V by considering the linear functional on $\mathscr{S}(V)$;

$$f(x) \mapsto \Phi_i(s,f) = \int |P(x)|_i^s f(x)\,dx, \tag{1.15}$$

with $f(x) \in \mathscr{S}(V)$. Then $\Phi_i(s,f)$ is a holomorphic function for $\mathrm{Re}(s) > -1$ and it is continued to the whole complex plane $s \in \mathbb{C}$ as a meromorphic function. In fact, by using the well-known equation

$$c\cdot b(s)\cdot\Phi_i(s,f) = \Phi_i\left(s+1, Q\left(-\frac{\partial}{\partial x}\right) f(x)\right), \qquad (c\text{: a constant}) \tag{1.16}$$

iteratedly, we have the meromorphic extension to the whole $s \in \mathbb{C}$ of $\Phi_i(s,f)$ for each $f(x) \in \mathscr{S}(V)$ (see, for example, p. 137 of [Sm-Sh] and Proposition 1.3 of [Sh]). We also denote it by $\Phi_i(s,f)$ for any $s \in \mathbb{C}$. Thus we have the linear functional $f(x) \mapsto \Phi_i(s,f)$ on $f(x) \in \mathscr{S}(V)$ for any $s \in \mathbb{C}$ except for poles, which is apparently a tempered distribution with a meromorphic parameter $s \in \mathbb{C}$ by the definition. We also denote by $|P(x)|_i^s$ the tempered distribution obtained by the analytic continuation of (1.15) to every $s \in \mathbb{C}$.

It is easily checked that the locations of poles of $|P(x)|_i^s$ are contained in the set

$$\mathrm{Cri}(P(x)^s) = \{s \in \mathbb{C};\, b(s+k) = 0 \text{ for some non-negative integer } k\}, \tag{1.17}$$

where $b(s)$ is the b-function for $P(x)$ by seeing the procedure of the analytic continuation through the formula (1.16). We call the elements of the set $\mathrm{Cri}(P(x)^s)$ *the critical points for* $P(x)^s$. More precisely, we have the following propositions. They may be well known and are easily verified only utilizing Eq. (1.16) repeatedly.

PROPOSITION 1.2. (1) *The tempered distribution with a meromorphic parameter s in* $\mathbb{C}$,

$$\left(\prod_{j=1}^{d} \Gamma(s+s_j)\right)^{-1} \cdot |P(x)|_i^s \qquad (i=1, \ldots, l), \tag{1.18}$$

is a tempered distribution on V with a holomorphic parameter $s \in \mathbb{C}$. *Here,* $-s_j$ $(j=1, \ldots, d)$ *are the roots of the b-function* $b(s)$ *for* $P(x)$ *defined in* (1.5).

(2) *The tempered distribution* $|P(x)|_i^s$ *is a relatively invariant tempered distribution corresponding to the character* $\chi(g)^s$ *whose support is contained in* $\bar{V}_i$. *However, when* $|P(x)|_i^s$ *has a pole at a critical point* s_0, *we take the coefficient of the lowest order term in the Laurent expansion given by* (1.13) *for* $|P(x)|_i^s$. *In particular,* $|P(x)|_i^s$ *is a* G^1*-invariant tempered distribution.*

2. A NECESSARY AND SUFFICIENT CONDITION FOR THE MAIN PROBLEM

In this section, we always assume that $(G_{\mathbb{C}}, \rho, V_{\mathbb{C}})$ is a regular prehomogeneous vector space satisfying the condition (A) and we continue to use the same notations and notions as in the previous section. The purpose of this section is to prove Theorem 2.7: a necessary and sufficient condition for the main problem given in the Introduction.

2.1. *Holonomic Systems* $\mathfrak{M}_s$ *and* $\mathfrak{M}_{(m)}$

Let $\mathscr{G}_{\mathbb{C}}$ be the complex Lie algebra of the complex linear algebraic group $G_{\mathbb{C}}$. Let $d\rho$ and $\delta\chi$ be the infinitesimal representation of ρ and χ, respectively. Similarly, we let $\mathscr{G}_{\mathbb{C}}^1$ be the complex Lie algebra of the subgroup $G_{\mathbb{C}}^1$. Consider the following systems of linear differential equations with one unknown function $u(x)$ on $V_{\mathbb{C}}$, where $s \in \mathbb{C}$ and $m \in \mathbb{N}$:

$$\mathfrak{M}_s; \qquad \left(\left\langle d\rho(A)\cdot x, \frac{\partial}{\partial x}\right\rangle - s\delta(A)\right) u(x) = 0, \qquad \text{for all } A \in \mathscr{G}_{\mathbb{C}}. \tag{2.1}$$

$$\mathfrak{M}_{(m)}; \qquad \begin{cases} \left\langle d\rho(A)\cdot x, \dfrac{\partial}{\partial x}\right\rangle u(x) = 0, & \text{for all } A \in \mathscr{G}_{\mathbb{C}}^1, \\ P(x)^m u(x) = 0. \end{cases} \tag{2.2}$$

Here, s is a complex number and m is a positive integer.

We shall review some notions on the system of linear differential equations, following [K2], for later use. Let

$$\mathfrak{M}; \qquad P_i\left(x, \frac{\partial}{\partial x}\right) u(x) = 0 \qquad (i = 1, ..., k)$$

be a system of linear differential equations with one unknown function $u(x)$. Here $P_i(x, \partial/\partial x)$'s $(i = 1, ..., k)$ are linear differential operators on $V_{\mathbb{C}}$ with holomorphic coefficients. Following [K2], we say that $\{P_i(x, \partial/\partial x)\}_{1 \leqslant i \leqslant k}$ is an *involutive basis* if $P_i(x, \partial/\partial x)$'s are closed under commutations. We define the *characteristic variety* $\mathrm{ch}(\mathfrak{M})$ as the common zeros of the principal symbols $\sigma(P_i)(x, y)$ in the conormal bundle $T^* V_{\mathbb{C}}$. In particular, we say that $\mathfrak{M}$ is a *holonomic system* if the dimension of $\mathrm{ch}(\mathfrak{M})$ coincides with that of $V_{\mathbb{C}}$. We often identify $T^* V_{\mathbb{C}}$ and $V_{\mathbb{C}} \times V_{\mathbb{C}}^*$.

Since the differential operators

$$\left\langle d\rho(A)\, x, \frac{\partial}{\partial x}\right\rangle - s\, \delta\chi(A), \qquad \text{with } A \in \mathscr{G}_{\mathbb{C}},$$

form an involutive basis for the system $\mathfrak{M}_s$, we have that the characteristic variety of $\mathfrak{M}_s$ is written as

$$\mathrm{ch}(\mathfrak{M}_s) = \{(x, y) \in V_{\mathbb{C}} \times V_{\mathbb{C}}^*;\ \langle d\rho(A)\, x, y\rangle = 0 \text{ for all } A \in \mathscr{G}_{\mathbb{C}}\}. \tag{2.3}$$

Similarly, the differential operators

$$\left\langle d\rho(A)\, x, \frac{\partial}{\partial x}\right\rangle \qquad \text{with } A \in \mathscr{G}_{\mathbb{C}}^1 \quad \text{and} \quad P(x)^m$$

commute with one another and hence they form an involutive basis for $\mathfrak{M}_{(m)}$, so the characteristic variety $\mathrm{ch}(\mathfrak{M}_{(m)})$ is written as

$$\begin{aligned}&\mathrm{ch}(\mathfrak{M}_{(m)})\\ &\quad = \{(x, y) \in V_{\mathbb{C}} \times V_{\mathbb{C}}^*;\ \langle d\rho(A)\, x, y\rangle = 0 \text{ for all } A \in \mathscr{G}_{\mathbb{C}}^1 \text{ and } P(x)^m = 0\}.\end{aligned} \tag{2.4}$$

PROPOSITION 2.1. *Under the condition* (A), *the systems of linear differential equations* $\mathfrak{M}_s$ *and* $\mathfrak{M}_{(m)}$ *are holonomic systems on* $V_{\mathbb{C}}$. *In particular, the characteristic varieties* $\mathrm{ch}(\mathfrak{M}_s)$ and $\mathrm{ch}(\mathfrak{M}_{(m)})$ are given by

$$\mathrm{ch}(\mathfrak{M}_s) = (V_{\mathbb{C}} \times \{0\}) \cup \left(\bigcup_{\alpha} T_{S_\alpha}^* V_{\mathbb{C}}\right), \tag{2.5}$$

$$\mathrm{ch}(\mathfrak{M}_{(m)}) = \bigcup_{\alpha} T_{S_\alpha}^* V_{\mathbb{C}}, \tag{2.6}$$

where $\{S_\alpha\}$ *is the set of* $G^1_{\mathbb{C}}$*-orbits in the singular set* $S_{\mathbb{C}}$ *and* $T^*_{S_\alpha} V_{\mathbb{C}}$ *stands for the conormal bundle of the orbit* S_α.

Proof. In the case of $\mathfrak{M}_s$, since $V_{\mathbb{C}}$ decomposes into a finite number of $G_{\mathbb{C}}$-orbits by the assumption (A), we can apply the arguments of Theorem 5.1.2 in [K2] directly. From the assumptions (A)(1), (2), we may let $(V_{\mathbb{C}} - S_{\mathbb{C}}) \cup S_1 \cup \cdots \cup S_k = V_{\mathbb{C}}$ be the decomposition of $V_{\mathbb{C}}$ into $G_{\mathbb{C}}$-orbits. By noting that $C_{\mathbb{C}} \times \{0\}$ is the closure of the conormal bundle of the orbit $V_{\mathbb{C}} - S_{\mathbb{C}}$, we have

$$\mathrm{ch}(\mathfrak{M}_s) \subset (V_{\mathbb{C}} \times \{0\}) \cup \left(\bigcup_k T^*_{S_k} V_{\mathbb{C}} \right), \tag{2.7}$$

by Theorem 5.1.2 in [K2]. The inverse inclusion relation of (2.7) is proved by checking that $\langle d\rho(A)\, x, y \rangle$ vanishes on the right-hand side of (2.7) for any $A \in \mathscr{G}_{\mathbb{C}}$. From the assumption (A), each singular orbit S_k is a $G_{\mathbb{C}}$-orbit, and hence we have (2.5).

In the case of $\mathfrak{M}_{(m)}$, the characteristic variety $\mathrm{ch}(\mathfrak{M}_{(m)})$ is the intersection of

$$\{(x, y) \in V_{\mathbb{C}} \times V^*_{\mathbb{C}}\,;\, P(x) = 0\} \tag{2.8}$$

and

$$\{(x, y) \in V_{\mathbb{C}} \times V^*_{\mathbb{C}};\, \langle d\rho(A)\, x, y \rangle = 0 \text{ for all } A \in \mathscr{G}^1_{\mathbb{C}}\}, \tag{2.9}$$

by (2.4). Suppose that (x_0, y_0) be a point in (2.8). Moreover, if (x_0, y_0) is in the set (2.9), then (x_0, y_0) is a point in the conormal bundle of the orbit $\rho(G^1_{\mathbb{C}}) \cdot x_0$, and the converse is true. Therefore, the intersection of the sets (2.8) and (2.9) coincides with the union of all the conormal bundles $T^*_{S_\alpha} V_{\mathbb{C}}$ of the $G^1_{\mathbb{C}}$-orbits S_α in the singular set $S_{\mathbb{C}}$. Thus we have (2.6). Q.E.D.

2.2. *Solutions to the Holonomic Systems* $\mathfrak{M}_s$ *and* $\mathfrak{M}_{(m)}$

We take a real form $(G^+_{\mathbb{R}}, \rho, V)$ of $(G_{\mathbb{C}}, \rho, V_{\mathbb{C}})$ and consider the hyperfunction solution space to the holonomic systems $\mathfrak{M}_s$ and $\mathfrak{M}_{(m)}$. We denote by $\mathscr{S}\!o\!\ell(\mathfrak{M}_s)$ and $\mathscr{S}\!o\!\ell(\mathfrak{M}_{(m)})$ the *hyperfunction solution space on* V *to the holonomic systems* $\mathfrak{M}_s$ *and* $\mathfrak{M}_{(m)}$, respectively. Then, as a consequence of a distinguished property on holonomic systems, we have the following proposition:

PROPOSITION 2.2. *The hyperfunction solution spaces* $\mathscr{S}\!o\!\ell(\mathfrak{M}_s)$ *and* $\mathscr{S}\!o\!\ell(\mathfrak{M}_{(m)})$ *are finite dimensional complex vector spaces.*

This is only a special case of the well-known and fundamental result on holonomic systems by Kashiwara, i.e., the finite dimensionality of the solution spaces of holonomic systems, and the proof is given in Chapter 5, Theorem 5.1.7, of [K2] by using cohomological language under a more general situation. In his theorem, $\mathscr{E}xt^0_{\mathscr{D}}(\mathfrak{M}, \mathscr{B}_M)_x$ means the hyperfunction solution space of the holonomic system $\mathfrak{M}$ at $x \in M$, where M is a real analytic manifold. He proved that $\dim \mathscr{E}xt^i_{\mathscr{D}}(\mathfrak{M}, \mathscr{B}_M)_x < +\infty$ for any $i \in \mathbb{N}^+ \cup \{0\}$.

We denote by $\mathscr{G}$ and $\mathscr{G}^1$ the real Lie algebras of $G^+_{\mathbb{R}}$ and G^1, respectively. Then, since $\mathscr{G}_{\mathbb{C}} = \mathscr{G} + \sqrt{-1}\mathscr{G}$, the holonomic system $\mathfrak{M}_s$ coincides with the following equation on $V_{\mathbb{R}}$:

$$\left(\left\langle d\rho(A)\cdot x, \frac{\partial}{\partial x}\right\rangle - s\delta(A)\right)\cdot u(x) = 0, \qquad \text{for all } A \in \mathscr{G}. \tag{2.10}$$

In fact, for any element $A \in \mathscr{G}_{\mathbb{C}}$, we can write it as $A = A_1 + \sqrt{-1}A_2$ by using the elements A_1 and A_2 in $\mathscr{G}$. We have

$$\begin{aligned}
&\left(\left\langle d\rho(A)\cdot x, \frac{\partial}{\partial x}\right\rangle - s\,\delta\chi(A)\right)\\
&\quad = \left(\left\langle d\rho(A_1)\cdot x, \frac{\partial}{\partial x}\right\rangle - s\,\delta\chi(A_1)\right)\\
&\qquad + \sqrt{-1}\left(\left\langle d\rho(A_2)\cdot x, \frac{\partial}{\partial x}\right\rangle - s\,\delta\chi(A_2)\right),
\end{aligned}$$

and hence if $u(x)$ is the solution to (2.10), then $u(x)$ is the solution to the holonomic system $\mathfrak{M}_s$. In the same way, we can show that the holonomic system $\mathfrak{M}_{(m)}$ coincides with the following system of differential equations:

$$\begin{cases} \left\langle d\rho(A)\cdot x, \dfrac{\partial}{\partial x}\right\rangle u(x) = 0, & \text{for all } A \in \mathscr{G}^1, \\ P(x)^m\, u(x) = 0. \end{cases} \tag{2.11}$$

PROPOSITION 2.3. *Let $u(x)$ be a relatively invariant tempered distribution on V corresponding to the character $\chi(g)^\lambda$ with $\lambda \in \mathbb{C}$. Then $u(x)$ is a solution to the holonomic system $\mathfrak{M}_\lambda$ and the converse is true.*

Proof. Let $u(x)$ be a tempered distribution satisfying

$$u(\rho(g)\,x) = \chi(g)^\lambda\, u(x), \tag{2.12}$$

for all $g \in G^+_{\mathbb{R}}$. For an element $A \in \mathscr{G}$ and a sufficiently small $t \in \mathbb{R}$,

$$\exp(tA) = \sum_{n=0}^{\infty} \frac{1}{n!}(tA)^n, \tag{2.13}$$

is well defined and is an element of $G_{\mathbb{R}}^{+}$. Then, we have

$$u(\rho(\exp(tA))\,x) = \chi(\exp(tA))^{\lambda} \cdot u(x). \tag{2.14}$$

By differentiating (2.14) at $t=0$, we get

$$\left\langle d\rho(A)\cdot x, \frac{\partial}{\partial x}\right\rangle u(x) = \lambda\, \delta\chi(A)\cdot u(x). \tag{2.15}$$

Since the holonomic system $\mathfrak{M}_{\lambda}$ is equivalent to Eq. (2.10) at $s=\lambda$, $u(x)$ is a solution to the holonomic system $\mathfrak{M}_{\lambda}$. Conversely, suppose that $u(x)$ is a tempered distribution to the holonomic system $\mathfrak{M}_{\lambda}$. Then we have (2.15) for any $A \in \mathscr{G}$. It is not difficult to prove (2.14) for a sufficiently small $t \in \mathbb{R}$ from (2.15). Any element $g \in G_{\mathbb{R}}^{+}$ is obtained as a finite product of the elements in $G_{\mathbb{R}}^{+}$ of the form (2.13). Thus we have (2.12) for all elements $g \in G_{\mathbb{R}}^{+}$. Q.E.D.

On the other hand, for solutions to $\mathfrak{M}_{(m)}$, we get the following proposition:

PROPOSITION 2.4. *Any G^1-invariant singular tempered distribution $u(x)$ on V belongs to the solution space $\mathscr{Sol}(\mathfrak{M}_{(m)})$ for a sufficiently large $m \in \mathbb{N}^{+}$. Conversely, if $u(x)$ is a tempered distribution in $\mathscr{Sol}(\mathfrak{M}_{(m)})$, then $u(x)$ is a G^1-invariant singular tempered distribution.*

Remark. In the process of the proof of Proposition 2.6, it will be proved that any hyperfunction solution to $\mathfrak{M}_{(m)}$ is a tempered distribution as a matter of fact. Therefore, by Proposition 2.4, we have shown that any $u(x)$ in $\mathscr{Sol}(\mathfrak{M}_{(m)})$ is a G^1-invariant singular tempered distribution.

Proof. Let $u(x)$ be a G^1-invariant singular tempered distribution on V. By the same arguments as in the proof of Proposition 2.3, the G^1-invariance of $u(x)$ means that $u(x)$ is the solution to the system of differential equations

$$\left\langle d\rho(A)\cdot x, \frac{\partial}{\partial x}\right\rangle u(x) = 0, \qquad \text{for all } A \in \mathscr{G}^1. \tag{2.16}$$

Therefore it remains to be shown that there exists a positive integer m such that $P(x)^m \cdot u(x) = 0$ if the support of $u(x)$ is contained in the singular set S.

First, note that the following well-known result on tempered distributions whose support is contained in a subvariety in V. Let $T(x)$ be a tempered distribution defined on V. Consider the compactification $\bar{V}$ of V by adding the point of infinity $\{\infty\}$: $\bar{V} = V \cup \{\infty\} \cong S^n$, where S^n stands for an n-dimensional sphere with $n := \dim V$. We may regard $T(x)$ as the restriction of a distribution $\tilde{T}(x)$ on $\bar{V}$: $\tilde{T}(x)\,|_{V} = T(x)$. Let U be a relatively compact domain in $\bar{V}$ and suppose that the support $\operatorname{supp}(\tilde{T}(x))$ is a non-singular subvariety M in U. Let $A(x)$ be a real analytic function which vanishes on M. Then there exists a positive integer q such that $A(x)^q \cdot \tilde{T}(x) \equiv 0$ on U. It is easily proved by applying, for example, Théorème 21 and Théorème 37 in Chapitre 3 of Schwartz' book [Sch].

Next, note that the singular set S can be decomposed into the disjoint union $S = S_1 \cup S_2 \cup \cdots \cup S_k$ such that $S^j = \bigcup_{i \geqslant j} S_i$ is an algebraic set in V and S_j is the non-singular locus of S^j. In fact, first, putting S_1 the non-singular locus of $S_0 := S$, then S_1 is an open dense subset of S. Next, letting S_2 be the non-singular locus of $S_0 - S_1$, and, in general, letting S_{i+1} be the non-singular locus of $S_i - S_{i-1}$ $(i \geqslant 1)$ and continuing this procedure, we obtain the above-mentioned disjoint decomposition of S.

Applying the Schwartz result, there exists a positive integer m_1 such that $\operatorname{supp}(P(x)^{m_1} \cdot u(x)) \subset S^2 = S_2 \cup \cdots \cup S_k$, for $P(x)$ vanishes on S and S_1 is the non-singular subvariety in V whose closure is compact in $\bar{V}$. Next, we get a positive integer m_2 such that $\operatorname{supp}(P(x)^{m_2} \cdot u(x)) \subset S^3$. Repeating this procedure, we get a positive integer m such that $\operatorname{supp}(P(x)^m \cdot u(x)) = \phi$ at last, that is to say, we have $P(x)^m \cdot u(x) = 0$. The converse is apparently true because $u(x) = 0$ on $V - S$ if $P(x)^m \cdot u(x) = 0$ on V with some positive integer m. Q.E.D.

Remark. The proof of this proposition has been given in Lemma 1.3 of [Sh] in a different way.

2.3. *Linear Combinations of Tempered Distributions with a Meromorphic Parameter*

Recall that $|P(x)|_i^s$ is a tempered distribution with a meromorphic parameter $s \in \mathbb{C}$ defined in (1.14). Then it is clear that $|P(x)|_i^\lambda$ is a relatively invariant tempered distribution corresponding to the character $\chi(g)^\lambda$ for any $\lambda \notin \operatorname{Cri}(P(x)^s)$. Therefore, by Proposition 2.3, $|P(x)|_i^\lambda$ is a solution to the holonomic system $\mathfrak{M}_\lambda$ for any $\lambda \notin \operatorname{Cri}(P(x)^s)$. Since $\mathfrak{M}_\lambda$ is a system of linear differential equations, any linear combination of $|P(x)|_i^\lambda$ is a solution to the holonomic system $\mathfrak{M}_\lambda$ for a non-critical $\lambda \in \mathbb{C}$. However, for a complex number λ in $\operatorname{Cri}(P(x)^s)$, we must reinterpret the meaning of the linear combination of $|P(x)|_i^s$ at $s = \lambda$. Namely, let $u_1(s, x), \ldots, u_l(s, x)$ be tempered distributions on V with a meromorphic parameter s near $s = \lambda$. We say that $u(x)$ can be written as *a linear combination of* $u_1(s, x), \ldots, u_l(s, x)$ *at* $s = \lambda$ if there exist meromorphic functions $c_1(s), \ldots, c_l(s)$ defined near $s = \lambda$ such that $\sum_{i=1}^{l} c_i(s) \cdot u_i(s, x)$ is *holomorphic* at $s = \lambda$ and $u(x)$ is given by

$$u(x) = \sum_{i=1}^{l} c_i(s) \cdot u_i(s, x) \,|_{s=\lambda}. \tag{2.17}$$

Following this definition, when we say that $u(x)$ is written as a *linear combination of* $|P(x)|_1^s, \ldots, |P(x)|_l^s$ at $s=\lambda$ with $\lambda \in \mathbb{C}$, this means that there exist meromorphic functions $c_1(s), \ldots, c_l(s)$ defined near $s=\lambda$ such that $\sum_{i=1}^l c_i(s) \cdot |P(x)|_i^s$ is holomorphic at $s=\lambda$ and $u(x) = \sum_{i=1}^l c_i(s) \cdot |P(x)|_i^s |_{s=\lambda}$. By the analytic continuation, it is clear that $u(x)$ thus obtained is a relatively invariant tempered distribution corresponding to the character $\chi(g)^\lambda$. Thus we have the following proposition from Proposition 2.3:

PROPOSITION 2.5. *Any linear combination of* $|P(x)|_i^s$ $(i=1, \ldots, l)$ *in the sense of* (2.17) *at* $s=\lambda$ *is a solution to the holonomic system* $\mathfrak{M}_\lambda$.

Remark. If $\lambda \notin \mathrm{Cri}(P(x)^s)$, then by the definition (2.17), $u(x)$ may be written as a linear combination of $|P(x)|_i^\lambda$ $(i=1, \ldots, l)$. However, it should be noted that $u(x)$ may not be expressed as a linear combination of the lowest order coefficients in the Laurent expansions of $|P(x)|_i^s$ $(i=1, \ldots, l)$ at $s=\lambda$ if $\lambda \in \mathrm{Cri}(P(x)^s)$.

2.4. *Some Properties of* $|P(x)|_i^s$

Let λ be a complex number. Then, from Proposition 1.1(2), (3), we have the Laurent expansion

$$|P(x)|_i^s = \sum_{j=0}^{\infty} T_{i\,\lambda}^{\,j}(x) \cdot (s-\lambda)^{j-j_{i,\lambda}} \tag{2.18}$$

at $s=\lambda$, where $j_{i,\lambda}$ is the order of the pole of $|P(x)|_i^s$ at $s=\lambda$. In particular, if $\lambda \notin \mathrm{Cri}(P(x)^s)$, then $j_{i,\lambda}=0$, and hence (2.18) is a Taylor expansion of $|P(x)|_i^s$ at $s=\lambda$. For a complex number λ, we let

$$\begin{aligned} V(\lambda) = {} & \text{the complex vector space generated by} \\ & \{T_{i\,\lambda}^{\,j}(x) \in \mathscr{S}'(V);\, i=1, \ldots, l, \text{ and } j=0, 1, \ldots\}. \end{aligned} \tag{2.19}$$

Though the following propositions are simple facts, they will play important roles in the proofs of Theorem 2.7.

PROPOSITION 2.6. *Let* $T_{i\,\lambda}^{\,j}(x)$ *be the tempered distribution defined in* (2.18) *and let* $j_{i,\lambda}$ *be the order of the pole of* $|P(x)|_i^s$ *at* $s=\lambda$.

(1) *Let* λ *be a point in* $\mathbb{C}$. *Then each tempered distribution* $T_{i\,\lambda}^{\,j}(x)$ *in* (2.18) *is* G^1*-invariant and the support of* $T_{i\,\lambda}^{\,j}(x)$ *is contained in the closure* $\bar{V}_i$. *For any* $\lambda \in \mathbb{C}$, *we have* $\mathrm{supp}(T_{i\,\lambda}^{\,j}(x)) = \bar{V}_i$ *and* $T_{i\,\lambda}^{\,j}(x)|_{V_i} = ((j-j_{i,\lambda})!)^{-1} \cdot |P(x)|^\lambda \log(|P(x)|)^{j-j_{i,\lambda}}$ *for* $j \geqslant j_{i,\lambda}$. *In particular, if* $\lambda \in \mathrm{Cri}(P(x)^s)$, *then the support of* $T_{i\,\lambda}^{\,j}(x)$ *is contained in* $\bar{V}_i \cap S$ *for* $j < j_{i,\lambda}$, i.e., *it is a* G^1*-invariant singular tempered distribution.*

(2) *We have*

$$\left(\delta\chi(I)^{-1}\left\langle x,\frac{\partial}{\partial x}\right\rangle-\lambda\right)T_i{}^j{}_\mu(x)=T_i{}^{j-1}{}_\mu(x)+(\mu-\lambda)\,T_i{}^j{}_\mu(x),\quad(2.20)$$

for any λ and μ in $\mathbb{C}$, and $j=0,1,2,\ldots$. However, we let $T_i{}^{-1}{}_\lambda(x)=0$. In particular, by letting $\lambda=\mu$, we have

$$\left(\delta\chi(I)^{-1}\left\langle x,\frac{\partial}{\partial x}\right\rangle-\lambda\right)\cdot T_i{}^j{}_\lambda(x)=T_i{}^{j-1}{}_\lambda(x).\quad(2.21)$$

(3) *If $\mu_1,\ldots,\mu_p$ are mutually different complex numbers, then $V(\mu_i)$'s are linearly independent, i.e., if $\sum_{i=1}^{p}a_iu_i=0$ with $a_i\in\mathbb{C}$ and $u_i\in V(\mu_i)$, $u_i\neq 0$, then a_i's $(i=1,\ldots,p)$ are all zero.*

(4) *All the $T_i{}^j{}_\lambda(x)$'s with $i=1,\ldots,l$, $j=0,1,2,\ldots$, and $\lambda\in\mathbb{C}$ are not zero. For fixed i in $1\leqslant i\leqslant l$ and $\lambda\in\mathbb{C}$, $T_i{}^k{}_\lambda(x)$ $(k=1,2,\ldots)$ are linearly* independent. *For a fixed $j\in\mathbb{N}^+$ and $\lambda\in\mathbb{C}$, we put $V_j(\lambda)$ the complex vector space generated by $T_i{}^j{}_\lambda(x)$ $(i=1,\ldots,l)$. Then, for a fixed $\lambda\in\mathbb{C}$, $V_j(\lambda)$'s $(j\in\mathbb{N}^+)$ are linearly independent.*

Proof. (1) The G^1-invariance of $T_i{}^j{}_\lambda(x)$ is directly implied by the G^1-invariance of $|P(x)|_i^s$ from the definition.

We note that $\int f(x)\cdot|P(x)|_i^s\,dx=0$ if $f(x)\in C_0^\infty(V-V_i)$ for any $s\in\mathbb{C}$. Then, from the definition of $T_i{}^j{}_\lambda(x)$, it is apparent that

$$\begin{aligned}&\int f(x)\,T_i{}^j{}_\lambda(x)\,dx\\&=\frac{1}{j!}\left(\frac{\partial}{\partial s}\right)^j(s-\lambda)^{j_{i,\lambda}}\int f(x)\,|P(x)|_i^s\,dx\,|_{s=\lambda}\\&=0,\end{aligned}\quad(2.22)$$

for any $f(x)\in C_0^\infty(V-V_i)$. Thus, we have

$$\operatorname{supp}(T_i{}^j{}_\lambda(x))\subset\bar{V}_i.\quad(2.23)$$

We suppose that $\lambda\in\mathrm{Cri}(P(x)^s)$. We shall show that $\operatorname{supp}(T_i{}^j{}_\lambda(x))\subset\bar{V}_i\cap S$ if $0\leqslant j\leqslant j_{i,\lambda}$. We may suppose that $j_{i,\lambda}>0$. Note that, for any $f(x)\in C_0^\infty(V_i)$, $\int f(x)\cdot|P(x)|_i^s\,dx$ is a non-zero holomorphic function in s near $s=\lambda$. We have

$$\int f(x)\cdot T_i{}^0{}_\lambda(x)\,dx=(s-\lambda)^{j_{i,\lambda}}\cdot\int f(x)\cdot|P(x)|_i^s\,dx\,|_{s=\lambda}=0,\quad(2.24)$$

and hence we have $\mathrm{supp}(T_i{}^0{}_\lambda(x)) \subset S$. Then, from (2.23), we have

$$\mathrm{supp}(T_i{}^0{}_\lambda(x)) \subset S \cap \bar{V}_i. \tag{2.25}$$

In order to prove that $\mathrm{supp}(T_i{}^j{}_\lambda(x)) \subset S \cap \bar{V}_i$ for any $0 \leqslant j < j_{i,\lambda}$, we use an induction on j. We assume that (2.25) is valid for $0 \leqslant j < p$ with $p < j_{i,\lambda}$. Then, for any $f(x) \in C_0^\infty(V_i)$, we have

$$\begin{aligned}
&\int f(x) \cdot T_i{}^j{}_\lambda(x)\, dx \\
&= (s-\lambda)^{j_{i,\lambda}-p} \cdot \int f(x) \cdot \Big(|P(x)|_i^s - \sum_{j=0}^{p-1} (s-\lambda)^{j-j_{i,\lambda}} \cdot T_i{}^j{}_\lambda(x)\Big)\, dx \,\Big|_{s=\lambda} \\
&= (s-\lambda)^{j_{i,\lambda}-p} \cdot \int f(x) \cdot |P(x)|_i^s \, dx \,\Big|_{s=\lambda} \\
&= 0.
\end{aligned} \tag{2.26}$$

Thus, we have $\mathrm{supp}(T_i{}^p{}_\lambda(x)) \subset S \cap \bar{V}_i$ from (2.23) and (2.26). Then, by induction on p, we have

$$\mathrm{supp}(T_i{}^j{}_\lambda(x)) \subset S \cap \bar{V}_i, \qquad \text{for any } 0 \leqslant j < j_{i,\lambda}. \tag{2.27}$$

Next, we suppose that $\lambda \in \mathbb{C}$. We shall show that $\mathrm{supp}(T_i{}^j{}_\lambda(x)) = \bar{V}_i$ for any $j \geqslant j_{i,\lambda}$. Let $f(x) \in C_0^\infty(V_i)$. Note that, by (2.27), we have

$$\begin{aligned}
&\int f(x) \cdot T_i{}^{j_{i,\lambda}+m}{}_\lambda(x)\, dx \\
&\quad = \frac{1}{m!}\left(\frac{\partial}{\partial s}\right)^m \int f(x) \cdot \big(|P(x)|_i^s \\
&\quad\qquad - (\text{the principal part of } |P(x)|_i^s \text{ at } s=\lambda)\big)\, dx \,\Big|_{s=\lambda} \\
&\quad = \frac{1}{m!}\left(\frac{\partial}{\partial s}\right)^m \int f(x) \cdot |P(x)|_i^s \, dx \,\Big|_{s=\lambda} \\
&\quad = \int f(x) \cdot \left(\frac{1}{m!} |P(x)|_i^\lambda (\log |P(x)|)^m\right) dx,
\end{aligned}$$

and hence, we have

$$T_i{}^{j_{i,\lambda}+m}{}_\lambda(x) \,|_{V_i} = \frac{1}{m!} |P(x)|_i^\lambda (\log |P(x)|)^m \,|_{V_i}.$$

That is to say, $\mathrm{supp}(T_i{}^j{}_\lambda(x)) = \bar{V}_i$ for $j \geqslant j_{i,\lambda}$.

(2) Consider the Taylor expansion of $(s-\lambda)^{j_{i,\lambda}} \cdot |P(x)|_i^s$ at $s=\lambda$:

$$(s-\lambda)^{j_{i,\lambda}} \cdot |P(x)|_i^s = \sum_{j=0}^{\infty} T_i{}^j{}_\lambda(x) \cdot (s-\lambda)^j. \tag{2.29}$$

We define the *differential operator* $D(\lambda)$ with $\lambda \in \mathbb{C}$ as

$$D(\lambda) = \left(\delta\chi(I)^{-1} \left\langle x, \frac{\partial}{\partial x} \right\rangle - \lambda \right). \tag{2.30}$$

Since $D(\lambda)(s-\lambda)^{j_{i,\lambda}} \cdot |P(x)|_i^s = (s-\lambda)^{j_{i,\lambda}+1} \cdot |P(x)|_i^s$, we have

$$\begin{aligned}(s-\lambda)^{j_{i,\lambda}+1} \cdot |P(x)|_i^s &= \sum_{j=0}^{\infty} D(\lambda) \cdot T_i{}^j{}_\lambda(x) \cdot (s-\lambda)^j \\ &= \sum_{j=0}^{\infty} T_i{}^j{}_\lambda(x) \cdot (s-\lambda)^{j+1},\end{aligned} \tag{2.31}$$

by termwise differentiation. Comparing the Taylor coefficients in (2.31), we have (2.21). By exchanging μ and λ in (2.21), we have

$$\begin{aligned}D(\lambda)\, T_i{}^j{}_\mu(x) &= (D(\mu) + (\mu-\lambda)) \cdot T_i{}^j{}_\mu(x) \\ &= T_i{}^{j-1}{}_\mu(x) + (\mu-\lambda) \cdot T_i{}^j{}_\mu(x),\end{aligned}$$

from (2.21), and hence we have (2.20).

Before proving (3), we shall show (4).

(4) It is clear that $T_i{}^0{}_\lambda(x) \neq 0$ from the definition. By using the relation (2.21), we have $T_i{}^j{}_\lambda(x) \neq 0$ if $T_i{}^{j-1}{}_\lambda(x) \neq 0$. Thus we have $T_i{}^j{}_\lambda \neq 0$ for all $j \geq 0$, by induction on j.

Next, in order to prove the linear independence, we assume that

$$\sum_{j=0}^{p} a_j \cdot T_i{}^j{}_\lambda(x) = 0, \tag{2.32}$$

with $a_i \in \mathbb{C}$ and p a positive integer. Then, operating $D(\lambda)^p$ on (2.32), we have $a_p \cdot T_i{}^0{}_\lambda = 0$, and hence we have $a_p = 0$. If $a_p = 0$, then, by operating $D(\lambda)^{p-1}$ on (2.23), we have $a_{p-1} = 0$. Thus, by induction on p, we see that $a_0 = a_1 = a_2 = \cdots = a_p = 0$; this means that $T_i{}^j{}_\lambda(x)$ $(j = 0, 1, 2, \ldots)$ are linearly independent. Last, we assume that

$$\sum_{j=0}^{p} a_j \cdot T_{i_j}{}^j{}_\lambda(x) = 0,$$

with $a_j \in \mathbb{C}$, p a positive integer, and i_j $(j = 0, \ldots, p)$ are integers in $1 \leq i_j \leq l$.

As in the proof of the linear independence of $T_i^j{}_\lambda(x)$ $(j=0, 1, ...)$, we have $a_0 = a_1 = \cdots = a_p = 0$, and hence the vector spaces $V_j(\lambda)$'s are linearly independent.

(3) Before proving the proposition (3), we have to show the following lemma.

LEMMA 2.6.1. (1) *Let $\lambda \in \mathbb{C}$. For any non-zero $x_\lambda \in V(\lambda)$, there exists a non-negative integer p such that $D(\lambda)^p x_\lambda \neq 0$ and $D(\lambda)^{p+1} x_\lambda = 0$.*

(2) *If $\lambda \neq \mu$ and x_λ is a non-zero element in $V(\lambda)$, then $D(\mu)^q x_\lambda \neq 0$ for any $q \in \mathbb{N}^+$.*

Proof. Since any element in $V(\lambda)$ is written as a finite linear combination of the elements in $\{T_i^j{}_\lambda(x)\}$, (1) is clear from the formula (2.21). We may suppose that $x_\lambda \in V(\lambda)$ is written as

$$x_\lambda = \sum_{j=0}^{j_0} \sum_{i=1}^{l} b_{i,j} \cdot T_i^j{}_\lambda(x),$$

with $b_{i,j} \in \mathbb{C}$. Note that $\sum_{i=1}^{l} b_{i,j} \cdot T_i^j{}_\lambda(x) \in V_j(\lambda)$. Moreover, we may assume that $\sum_{i=1}^{l} b_{i,j_0} \cdot T_i^j{}_\lambda(x) \neq 0$. By the formula (2.20), $D(\mu)^q \cdot x_\lambda$ is expressed as

$$(\lambda-\mu)^q \sum_{i=1}^{l} b_{i,j_0} \cdot T_i^{j_0}{}_\lambda(x) + \{\text{a linear combination of the element}$$
$$\text{in } V_0(\lambda) \oplus V_1(\lambda) \oplus \cdots \oplus V_{j_0-1}(\lambda)\}.$$

Since the vector spaces $V_j(\lambda)$ $(j=0, ..., j_0)$ are linearly independent and since $(\lambda-\mu)^q \sum_{i=1}^{l} b_{i,j_0} \cdot T_i^{j_0}{}_\lambda(x) \neq 0$ for any $q \in \mathbb{N}^+$, we have $D(\lambda) \cdot x_\lambda \neq 0$ for any $q \in \mathbb{N}^+$. (Lemma 2.6.1, Q.E.D.)

We go back to the proof of (3). We suppose that

$$\sum_{i=1}^{p} a_i \cdot u_i = 0, \tag{2.33}$$

with $a_i \in \mathbb{C}$, and $u_i \in V(\mu_i)$, $u_i \neq 0$. We let q_1 be a positive integer such that $D(\mu_1)^{q_1+1} u_1 = 0$ and $D(\mu_1)^{q_1} u_1 \neq 0$ and let $q_1, ..., q_p$ be sufficiently large integers. By operating $D(\mu_1)^{q_1} \cdot D(\mu_2)^{q_2} \cdots D(\mu_p)^{q_p}$ on both sides of (2.33), we have

$$D(\mu_1)^{q_1} \cdot D(\mu_2)^{q_2} \cdots D(\mu_p)^{q_p} \cdot \sum_{i=1}^{p} a_i \cdot u_i = a_1 \cdot y_1 = 0,$$

with $y_1 \in V(\mu_1)$ and $y_1 \neq 0$. Thus we have $a_1 = 0$. In the same way, by taking an integer suitably, and taking sufficiently large integers $q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_p$, we have $a_i = 0$. Therefore, $u_1, \ldots, u_p$ are linearly independent if each u_i is not zero and is an element of $V(\mu_i)$. Hence $V(\mu_i)$'s are linearly independent. (Proposition 2.6, Q.E.D.)

Remark. In [Ru2], Rubenthaler obtained some interesting results closely related to Proposition 2.6.

2.5. *The Proof of the Main Theorem*

We let

$$V^{\mathrm{neg}}(\lambda) = \text{the complex vector space generated by}$$

$$\{T_i{}^j{}_\lambda(x); i = 1, \ldots, l, 0 \leqslant j < j_{i,\lambda}\}, \tag{2.34}$$

which is by definition the space of linear combinations of negative order Laurent coefficients in $|P(x)|_i^s$ with $i = 1, \ldots, l$ at $s = \lambda$. If $\lambda \notin \mathrm{Cri}(P(x)^s)$, then $V^{\mathrm{neg}}(\lambda) = \{0\}$. When $\lambda \in \mathrm{Cri}(P(x)^s)$ and $j_{i,\lambda}$ is positive, the vector space $V^{\mathrm{neg}}(\lambda) \neq \{0\}$ by Proposition 2.6(4) and any element of $V^{\mathrm{neg}}(\lambda)$ is a singular G^1-invariant tempered distribution. The main theorem we want to show in this section is to find a necessary and sufficient condition in order that any singular G^1-invariant tempered distribution is written as a finite linear combination of $V^{\mathrm{neg}}(\lambda)$ with $\lambda \in \mathrm{Cri}(P(x)^s)$. Namely, we have the following theorem.

THEOREM 2.7. *The following conditions* (B) *and* (C) *are equivalent under the condition* (A), (1), (2), (3).

(B) *Any G^1-invariant singular tempered distribution $u(x)$ is written as a finite linear combination of negative order Laurent coefficients of $|P(x)|_i^s$ at poles, i.e., $u(x)$ is an element of $V^{\mathrm{neg}} = \bigoplus_{\lambda \in \mathbb{C}} V^{\mathrm{neg}}(\lambda)$.*

(C) *For any $\lambda \in \mathbb{C}$, any relatively invariant tempered distribution corresponding to the character $\chi(g)^\lambda$ is obtained as a linear combination of $|P(x)|_i^s$ at $s = \lambda$ in the sense of* (2.17).

Proof. We begin with the proof of that (B) implies (C). From Proposition 2.3, we have only to show that any solution to $\mathfrak{M}_\lambda$ is written as a linear combination of $|P(x)|_i^s$ at $s = \lambda$ under the condition (B).

First, we have to show the following lemma:

LEMMA 2.7.1. *Let $\lambda \in \mathbb{C}$. For any $u(x) \in \mathcal{S}ol(\mathfrak{M}_\lambda)$, there exist complex numbers $a_1, \ldots, a_l$ such that*

$$u(x)\,|_{V-S} = \sum_{i=1}^{l} a_i \cdot T_i{}^{j_{i,\lambda}}{}_\lambda(x)\,|_{V-S}, \tag{2.35}$$

where $T_i^{j_{i,\lambda}}{}_\lambda(x)$ are the tempered distributions in (2.18) *and $j_{i,\lambda}$ is the order of $|P(x)|_i^s$ at $s=\lambda$.*

Proof. First, note that $\mathfrak{M}_\lambda$ is an elliptic system of linear differential equations on $V-S$ because the characteristic variety of $\mathfrak{M}_\lambda$ on $V-S$ is just the zero section $(V-S)\times\{0\}$ (Proposition 2.1). Therefore, from Sato's fundamental theorem, (see, for example, Chapter III, Theorem 3.4.3, [Ka-Kw-Ki]), $u(x)$ is a real analytic function on $V-S$.

Let x_i be a point in V_i $(i=1, ..., l)$, where V_i is a connected component defined in (1.7). Then, $u(x_i)$ is well defined because $u(x)$ is real analytic on V_i. Note that

$$u(\rho(g)\cdot x_i)=\chi(g)^\lambda\cdot u(x_i), \tag{2.36}$$

for any $g\in G_{\mathbb{R}}^+$ and that V_i is a single $G_{\mathbb{R}}^+$-orbit. Then since

$$\begin{aligned} u(\rho(g)\cdot x_i)&=(u(x_i)/|P(x_i)|^\lambda)\cdot\chi(g)^\lambda\cdot|P(x_i)|^\lambda\\ &=(u(x_i)/|P(x_i)|^\lambda)\cdot|P(\rho(g)\,x_i)|^\lambda, \end{aligned}$$

we have

$$u(x)\,|_{V_i}=a_i\cdot|P(x)|^\lambda\,|_{V_i}, \tag{2.37}$$

with a constant $a_i\in\mathbb{C}$.

Next note that

$$T_i^{j_{i,\lambda}}{}_\lambda(x)\,|_{V-S}=\begin{cases}|P(x)|^\lambda & \text{if}\quad x\in V_i,\\ 0 & \text{if}\quad x\notin V_i.\end{cases} \tag{2.38}$$

This follows from Proposition 2.6(1) and its proof. Therefore, by (2.37) and (2.38), we have

$$u(x)\,|_{V-S}=\sum_{i=0}^{l}a_i\cdot T_i^{j_{i,\lambda}}{}_\lambda(x)\,|_{V-S}.$$

(Lemma 2.7.1, Q.E.D.)

By Lemma 2.7.1, we have

$$v(x)=u(x)-\sum_{i=1}^{l}a_i\cdot T_i^{j_{i,\lambda}}{}_\lambda(x) \tag{2.39}$$

is a distribution whose support is contained in the singular set S. From the assumption (B), $v(x)$ is written as a finite linear combination of $\{T_i^j{}_\lambda(x)\}$ with $i=1, ..., l$, $0\leqslant j<j_{i,\lambda}$ and $\lambda\in\mathbb{C}$. Therefore, $u(x)$ is written as a finite linear combination of the Laurent coefficients of $|P(x)|_i^s$. Namely, there

exist mutually different complex numbers $\mu_1, ..., \mu_p$ such that $u(x)$ is an element of $V(\mu_1) \oplus V(\mu_2) \oplus \cdots \oplus V(\mu_p)$. However, we have the following general lemma.

LEMMA 2.7.2. *Let $u(x)$ be a distribution solution to $\mathfrak{M}_\lambda$ and suppose that $u(x)$ is an element of $V(\mu_1) \oplus \cdots \oplus V(\mu_p)$, where $\mu_1, ..., \mu_p$ are mutually different complex numbers. Then $u(x)$ is written as a linear combination of $|P(x)|_i^s$ at $s = \lambda$.*

Proof. We may assume that

$$u(x) = \sum_{k=1}^{p} \left(\sum_{i,j} a_i{}^j{}_k \cdot T_i{}^j{}_{\mu_k}(x) \right), \tag{2.40}$$

with $a_i{}^j{}_k \in \mathbb{C}$. Almost all $a_i{}^j{}_k$'s are zero. We shall prove:

SUBLEMMA 2.7.3. *Among the terms in the sum of* (2.40), *if $\mu_k \neq \lambda$, then we have*

$$\sum_{i,j} a_i{}^j{}_k \cdot T_i{}^j{}_{\mu_k}(x) = 0. \tag{2.41}$$

Proof. Since $D(\lambda)\, u(x) = 0$, we have

$$\sum_{k=1}^{p} \left(\sum_{i,j} a_i{}^j{}_k \cdot (T_i{}^{j-1}{}_{\mu_k}(x) + (\mu_k - \lambda) \cdot T_i{}^j{}_{\mu_k}(x)) \right) = 0,$$

by (2.20) and (2.40). Then, from Proposition 2.6(3), for each k, we get

$$\sum_{i,j} a_i{}^j{}_k \cdot (T_i{}^{j-1}{}_{\mu_k}(x) + (\mu_k - \lambda) \cdot T_i{}^j{}_{\mu_k}(x)) = 0. \tag{2.42}$$

Let j_0 be the largest integer among the j's in the sum of the left-hand side of (2.41). Then we may suppose that (2.41) is written in the form of $\sum_{i=1}^{l} \sum_{j=0}^{j_0} a_i{}^j{}_k \cdot T_i{}^j{}_{\mu_k}(x)$, and we suppose that

$$\sum_{i=1}^{l} a_i{}^{j_0}{}_k \cdot T_i{}^{j_0}{}_{\mu_k}(x) \neq 0. \tag{2.43}$$

Then we have

$$\begin{aligned}
&\text{The left-hand side of (2.42)} \\
&= \sum_{j=0}^{j_0-1} \left(\sum_{i=1}^{l} (a_i{}^{j+1}{}_k + (\mu_k - \lambda) \cdot a_i{}^j{}_k) \cdot T_i{}^j{}_{\mu_k}(x) \right) \\
&\quad + \left(\sum_{i=1}^{l} (\mu_k - \lambda) \cdot a_i{}^{j_0}{}_k \cdot T_i{}^{j_0}{}_{\mu_k}(x) \right) \\
&= 0,
\end{aligned}$$

and hence

$$\sum_{i=1}^{l} a_i^{j_0}{}_k \cdot T_i^{j_0}{}_{\mu_k}(x) = 0,$$

because of Proposition 2.6(4) and that $\mu_k - \lambda \neq 0$. This is the contradiction. Therefore there is no such j_0 satisfying (2.43). Then we have (2.41).

(Sublema 2.7.3, Q.E.D.)

If all the μ_k's are different from λ, then $u(x)$ vanishes identically from Sublemma 2.7.3, and hence Lemma 2.7.2 is proved.

We suppose that there exists μ_k which coincides with λ. Then, from Sublemma 2.7.3, $u(x)$ is expressed as

$$u(x) = \sum_{j=0}^{j_0} \left(\sum_{i=1}^{l} (a_i^j{}_\lambda \cdot T_i^j{}_\lambda(x) \right),$$

with some non-negative integer j_0. By operating $D(\lambda)$ on $u(x)$, we have

$$D(\lambda) \cdot u(x) = \sum_{j=0}^{j_0-1} \left(\sum_{i=1}^{l} (a_i^{j+1}{}_\lambda \cdot T_i^j{}_\lambda(x) \right) = 0.$$

From the linear independence of $V_j(\lambda)$'s $(j = 0, ..., j_0 - 1)$ (Proposition 2.6(4)), we get

$$\sum_{i=1}^{l} a_i^{j+1}{}_\lambda \cdot T_i^j{}_\lambda(x) = 0 \qquad (j = 0, ..., j_0 - 1). \tag{2.44}$$

Operating $D(\lambda)$ on (2.44) repeatedly, we have

$$\sum_{i=1}^{l} a_i^j{}_\lambda \cdot T_i^p{}_\lambda(x) = 0 \qquad (j = 1, ..., j_0) \tag{2.45}$$

for all $0 \leqslant p < j$.

Let $j_{i,\lambda}$ be the order of the pole of $|P(x)|_i^s$ at $s = \lambda$. Then, we have

$$(s-\lambda)^{j_{i,\lambda}} \cdot |P(x)|_i^s = \sum_{j=0}^{\infty} T_i^j{}_\lambda(x) \cdot (s-\lambda)^j.$$

We put

$$b_i(s) = \sum_{j=0}^{j_0} a_i^{j_0 - j} \cdot (s-\lambda)^j.$$

Then, using (2.45), it is easy to prove that $\sum_{i=1}^{l} b_i(s)\cdot(s-\lambda)^{j_{i,\lambda}}\cdot|P(x)|_i^s$ has the Taylor expansion:

$$(s-\lambda)^{j_0}\cdot u(x)+(\text{trems of degree more than } j_0+1),$$

which implies that

$$\sum_{i=1}^{l} b_i(s)\cdot(s-\lambda)^{-j_0+j_{i,\lambda}}\cdot|P(x)|_i^s\,|_{s=\lambda}=u(x). \tag{2.46}$$

Thus, from the definition, $u(x)$ is written as a linear combination of $|P(x)|_i^s$ at $s=\lambda$. (Lemma 2.7.2, Q.E.D.)

Therefore, any tempered distribution solution $u(x)$ to $\mathfrak{M}_\lambda$ is written as a linear combination of $|P(x)|_i^s$ at $s=\lambda$ in the sense of (2.17). Thus, from Proposition 2.3, we have proved that (B) implies (C).

Next we shall show that (C) yields (B). From Proposition 2.4, we have only to show that any solution to $\mathfrak{M}_{(m)}$ is an element of the vector space

$$V^{\text{neg}}=\bigoplus_{\lambda\in\text{Cri}(P(x)^s)} V^{\text{neg}}(\lambda), \tag{2.47}$$

for each $m\in\mathbb{N}^+$ under the condition (C).

LEMMA 2.7.4. *The differential operator* $D(0)=\delta(I)^{-1}\langle x,\partial/\partial x\rangle$ *is a* $\mathbb{C}$*-linear endomorphism on* $\mathcal{Sol}(\mathfrak{M}_{(m)})$.

Proof. Let $u(x)\in\mathcal{Sol}(\mathfrak{M}_{(m)})$. For any $A\in\mathcal{G}^1$, the operators $D(0)$ and $\langle d\rho(A)\,x,\partial/\partial x\rangle$ commute with each other. Then we have

$$\left\langle d\rho(A)\cdot x,\frac{\partial}{\partial x}\right\rangle\cdot D(0)\cdot u(x)=D(0)\cdot\left\langle d\rho(A)\,x,\frac{\partial}{\partial x}\right\rangle\cdot u(x)=0.$$

On the other hand, we have

$$P(x)^m\cdot D(0)\cdot u(x)=(D(0)\cdot P(x)^m-m\cdot P(x)^m)\cdot u(x)=0.$$

Thus we have $D(0)\,u(x)\in\mathcal{Sol}(\mathfrak{M}_{(m)})$. Since $D(0)$ is a linear differential operator, $D(0)$ is a $\mathbb{C}$-linear endomorphism on $\mathcal{Sol}(\mathfrak{M}_{(m)})$.

(Lemma 2.7.4, Q.E.D.)

From Proposition 2.2, the dimension of $\mathcal{Sol}(\mathfrak{M}_{(m)})$ is finite. We put p the dimension of $\mathcal{Sol}(\mathfrak{M}_{(m)})$. Then taking the basis of $\mathcal{Sol}(\mathfrak{M}_{(m)})$,

$$\{u_i(x)\}_{i=1,\dots,p}, \tag{2.48}$$

suitably, the linear operator $D(0)$ is written as a Jordan canonical form,

$$D(0)\cdot\begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_p(x) \end{bmatrix} = \begin{bmatrix} A_1, & & & \\ & A_2, & & 0 \\ & & \ddots & \\ & 0 & & A_q \end{bmatrix} \begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_p(x) \end{bmatrix}, \tag{2.49}$$

where

$$A_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & 0 & \ddots & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \qquad \text{with } \lambda_i \in \mathbb{C} \text{ and } i = 1, ..., q.$$

Here we let k_i be the size of the matrix A_i and $\sum_{i=1}^{q} k_i = p$.

Now our problem is reduced to the proof of

$$u_i(x) \in V^{\mathrm{neg}}, \tag{2.50}$$

for $i = 1, ..., p$.

Now we define the system of linear differential equations $\mathfrak{M}_\lambda^{(k)}$ with $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}^+$:

$$\mathfrak{M}_\lambda^{(k)}; \quad \begin{cases} \left(\left\langle x, \dfrac{\partial}{\partial x}\right\rangle - \lambda\, \delta\chi(I)\right)^k u(x) = 0, \\ \left\langle d\rho(A)\cdot x, \dfrac{\partial}{\partial x}\right\rangle u(x) = 0 \qquad \text{for all } A \in \mathscr{G}^1. \end{cases} \tag{2.51}$$

Then the system $\mathfrak{M}_\lambda^{(k)}$ is holonomic because $\mathrm{ch}(\mathfrak{M}_\lambda^{(k)}) = \mathrm{ch}(\mathfrak{M}_\lambda)$. In particular, $\mathfrak{M}_\lambda^{(1)} = \mathfrak{M}_\lambda$ from the definition. We consider the system of linear differential equations $\mathfrak{N}_\lambda^{(k)}$ with $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}^+$:

$$\mathfrak{N}_\lambda^{(k)}; \quad \begin{cases} \left\langle x, \dfrac{\partial}{\partial x}\right\rangle - \delta\chi(I) \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots & 1 \\ & & & \lambda \end{bmatrix} \begin{bmatrix} v_1(x) \\ v_2(x) \\ \vdots \\ v_k(x) \end{bmatrix} = 0, \\ \left\langle d\rho(A)\cdot x, \dfrac{\partial}{\partial x}\right\rangle {}^t(v_1(x), ..., v_k(x)) = 0 \qquad \text{for all } A \in \mathscr{G}^1. \end{cases} \tag{2.52}$$

The system $\mathfrak{N}_\lambda^{(k)}$ is a differential equation to the vector valued function $v(x) = {}^t(v_1(x), ..., v_k(x))$. The following lemma is trivial.

LEMMA 2.7.5. *For any solution* $v(x) = {}^t(v_1(x), ..., v_k(x))$ *to* $\mathfrak{N}_\lambda^{(k)}$, *all the components* $v_1(x), ..., v_k(x)$ *are solutions to* $\mathfrak{M}_\lambda^{(k)}$. *Conversely, for any solution* $v(x)$ *to* $\mathfrak{M}_\lambda^{(k)}$, *the vector valued function* $v(x) = {}^t(v(x), D(\lambda)\, v(x), ..., D(\lambda)^{k-1}\, v(x))$ *is a solution to* $\mathfrak{N}_\lambda^{(k)}$.

The next corollary follows from Eq. (2.49).

COROLLARY 2.7.6. *For any* $u_i(x)$ *in the basis* (2.48), *there exist* $\lambda \in \mathbb{C}$ *and* $k \in \mathbb{N}^+$ *such that* $u_i(x)$ *is a solution to* $\mathfrak{M}_\lambda^{(k)}$.

We shall examine the structure of $\mathcal{Sol}(\mathfrak{M}_\lambda^{(k)})$. We let

$$u_i(s, x) = (s - \lambda)^{j_{i,\lambda}} \cdot |P(x)|_i^s \qquad (i = 1, ..., l) \tag{2.53}$$

with $j_{i,\lambda}$ the order of the pole of $|P(x)|_i^s$ at $s = \lambda$. Then $u_i(s, x)$ is a tempered distribution with a holomorphic parameter s near $s = \lambda$. We let

$$u_i^{(j)}(s, x) = \left(\frac{\partial}{\partial s}\right)^j u_i(s, x). \tag{2.54}$$

LEMMA 2.7.7. *Under the assumption* (C), *any solution to* $\mathfrak{M}_\lambda^{(k)}$ *is given by a linear combination of*

$$u_i^{(j)}(s, x) \qquad \textit{with } i = 1, ..., l \textit{ and } j = 0, 1, ..., k-1, \tag{2.55}$$

at $s = \lambda$.

Proof. We shall show this by induction on k. It is true for $k = 1$ by the assumption (C). We suppose that this lemma is true for $1 \leqslant k \leqslant p$. Let $u(x)$ be a solution to $\mathfrak{M}_\lambda^{(p+1)}$. We let

$$v(x) = D(\lambda)^p \cdot u(x).$$

Then $v(x)$ is a solution to $\mathfrak{M}_\lambda^{(1)}$. Therefore, from the induction hypothesis, $v(x)$ is written as

$$v(x) = \sum_{i=1}^{l} a_i(s) \cdot u_i(s, x)\,|_{s=\lambda},$$

where $a_i(s)$'s are meromorphic functions near $s = \lambda$ and $\sum_{i=1}^{l} a_i(s) \cdot u_i(s, x)$ is holomorphic near $s = \lambda$.

Consider the function

$$w(s, x) = \left(u(x) - (1/p!) \sum_{i=1}^{l} a_i(s) \cdot u_i^{(p)}(s, x)\right).$$

Then we have

$$\begin{aligned}
&D(\lambda)^p\, w(s, x)\,|_{s=\lambda} \\
&\quad = D(\lambda)^p\, u(x) - \frac{1}{p!} \sum_{i=1}^{l} a_i(s) \left(\delta\chi(I)^{-1} \left\langle x, \frac{\partial}{\partial x} \right\rangle - \lambda \right)^p \\
&\qquad \times \left(\frac{\partial}{\partial s} \right)^p u_i(s, x)\,|_{s=\lambda} \\
&\quad = v(x) - \frac{1}{p!} \sum_{i=1}^{l} a_i(s) \left(\frac{\partial}{\partial s} \right)^p (s-\lambda)^p\, u_i(s, x)\,|_{s=\lambda} \\
&\quad = v(x) - \sum_{i=1}^{l} a_i(s)\, u_i(s, x)\,|_{s=\lambda} \\
&\quad = 0,
\end{aligned}$$

and hence $w(s, x)|_{s=\lambda}$ is a solution to $\mathfrak{M}_\lambda^{(p)}$ because it is clear that $w(s, x)$ is G^1-invariant.

Therefore, from the induction hypothesis, we have

$$w(s, x)\,|_{s=\lambda} = \sum_{i=1}^{l} \sum_{j=0}^{p-1} a_i^j(s) \cdot u_i^{(j)}(s, x)\,|_{s=\lambda}, \tag{2.56}$$

where $a_i^j(s)$'s are meromorphic functions near $s=\lambda$ and $\sum_{i=1}^{l} \sum_{j=0}^{p-1} a_i^j(s) \cdot u_i(s, x)$ is holomorphic near $s=\lambda$. Thus we have

$$\begin{aligned}
u(x) &= w(s, x) + \frac{1}{p!} \sum_{i=1}^{l} a_i(s) \cdot u_i^{(p)}(s, x)\,|_{s=\lambda} \\
&= \sum_{i=1}^{l} \left(\sum_{j=0}^{p-1} a_i^j(s) \cdot u_i^{(j)}(s, x) + a_i(s) \cdot u_i^{(p)}(s, x) \right)\Big|_{s=\lambda},
\end{aligned} \tag{2.57}$$

and hence $u(x)$ is given by a linear combination of $u_i^{(j)}(s, x)$ ($i = 1, ..., l$ and $j = 0, 1, ..., p$). Therefore this lemma is true for $k = p+1$. Thus by induction on k, Lemma 2.7.7 is proved. (Lemma 2.7.7, Q.E.D.)

COROLLARY 2.7.8. *The solution space* $\mathcal{Sol}(\mathfrak{M}_\lambda^{(k)})$ *is a subspace of* $V(\lambda)$.

Proof. It is clear from the definition.

LEMMA 2.7.9. *If* $u(x)$ *is a tempered distribution satisfying* $u(x) \in \mathcal{Sol}(\mathfrak{M}_\lambda^{(k)})$ *and* $\operatorname{supp}(u(x)) \subset S$, *then we have* $u(x) \in V^{\mathrm{neg}}(\lambda)$.

Proof. The tempered distribution $u(x)$ is written as

$$u(x) = \sum_{i=1}^{l} \sum_{j=0}^{j_0} a_i{}^j{}_\lambda \cdot T_i{}^j{}_\lambda(x), \tag{2.58}$$

with $a_i{}^j{}_\lambda \in \mathbb{C}$ and $T_i{}^j{}_\lambda(x)$ are tempered distributions defined in (2.18). Since $\mathrm{supp}(u(x)) \subset S$, we have

$$\int u(x)\, f(x)\, dx = 0, \tag{2.59}$$

for any $f(x) \in C_0^\infty(V_i)$. By (2.58) and Proposition 2.6(1), we have

$$\begin{aligned}
&\text{The left-hand side of (2.59)}\\
&= \int \sum_{j=0}^{j_0} a_i{}^j{}_\lambda \cdot T_i{}^j{}_\lambda(x)\, f(x)\, dx,\\
&= \int \sum_{j=j_{i,\lambda}}^{j_0} a_i{}^j{}_\lambda \cdot |P(x)|^\lambda (\log |P(x)|)^{j-j_{i,\lambda}} f(x)\, dx,\\
&= 0.
\end{aligned}$$

Since $|P(x)|^\lambda (\log |P(x)|)^m$ $(m = 0, 1. ...)$ are linearly independent functions on V_i, the complex numbers $a_i{}^j{}_\lambda$ with $j \geqslant j_{i,\lambda}$ are all zero. Thus we have $u(x) \in V^{\mathrm{neg}}(\lambda)$. (Lemma 2.7.9, Q.E.D.)

From Corollaries 2.7.6 and 2.7.8 and Lemma 2.7.9, any $u_i(x)$ in the basis (2.48) is an element of V^{neg} defined in (2.47). Therefore, any solution to $\mathfrak{M}_{(m)}$, which is written by the basis (2.48), is an element of V^{neg}. Thus we complete the proof of that (C) implies (B). (Theorem 2.7, Q.E.D.)

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